Open problem list for the Structural Graph Theory Workshop at Gułtowy 2019*

September 5, 2019

Open problems suggested by participants

1 Tighter bounds for linear colorings

Suggested by Marcin Pilipczuk

One of the alternative definitions of treedepth is via centered colorings. A centered coloring of a graph G is a coloring $c : V(G) \to \mathbb{Z}$ such that every connected subgraph of G has a vertex of unique color. The minimum possible number of colors in a centered coloring of a graph is the treedepth of the graph.
In [2] a notion of a linear coloring is studied, where only paths in G are required to have a vertex of a unique color. The authors prove that if k colors suffice for linear coloring, then $\widetilde{\mathcal{O}}(k^{190})$ colors suffice for a centered coloring. An improved bound in the excluded minor approximation for treedepth [1] improved it to $\widetilde{\mathcal{O}}(k^{19})$. Can we push it further? To get significant improvement, one would need to

1. Avoid the usage of the grid minor theorem in the proofs of [2, 1].
2. **(Essentially solved)** Improve the lower bound of the linear coloring number of a grid containing a k-wall from $\Omega(\sqrt{k})$ to $\Omega(k)$.
3. **(New)** The best known separation example gives $\chi_{\mathrm{lin}}(G) \sim \mathrm{td}(G)/2$. Make a larger separation.

Update 26.06 (Archontia, Terka, Irene, Marcin P.) Linear coloring number of a $k \times k$ grid is at least a quarter of the treedepth. Linear coloring number of a $k \times k$ subdivided wall is $\Omega(k/\log k)$. The latter gives $\widetilde{\mathcal{O}}(k^{10})$ bound for linear coloring vs treedepth in general graphs. Here k^9 comes from the grid minor theorem, so one really needs to avoid it in order to have any significant further improvement.

References

[1] W. Czerwinski, W. Nadara, and M. Pilipczuk. Improved bounds for the excluded-minor approximation of treedepth. *CoRR*, abs/1904.13077, 2019.

[2] J. Kun, M. P. O'Brien, and B. D. Sullivan. Treedepth bounds in linear colorings. *CoRR*, abs/1802.09665, 2018.

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2 Excluded minor pathwidth approximation

Posed by Kawarabayashi and Rossman [1].

(Solved) Does there exists a polynomial f such that every graph G of pathwidth at least $f(k)$ but treewidth less than k contains a subdivision of a binary tree of depth k as a subgraph? Kawarabayashi and Rossman [1] proved an analogous statement for treedepth, with a path of length 2^k being a third possible outcome.

Update 26.06 (Carla, Gwen, Wojtek, Bartek W) There exists a function $f(t, d) = \mathcal{O}(dt \log t)$ such that for every $d, t \geq 1$ and every graph G of pathwidth at least $f(t, d)$ has either treewidth at least t or contains T_d as a minor, where T_d is a maximal tree of height d with root having three children and all other internal nodes having two children each.
The proof gives $f(t, 1) = 3$ and $f(t, d + 1) = f(t, d) + t\lfloor \log t \rfloor + 6t$.

Update 27.06 (Carla, Gwen, Wojtek, Bartek W) Function got improved to $4dt$.

Update 27.06 (Marcin P) The same methodology proves that a graph of treewidth less than a and without tree of treedepth larger than b as a subgraph has treedepth bounded by $a(b - 1) + 1$.
However, the following conjectures are open. (For the first one, Gwen has a simple argument for Ca^2b.)

Conjecture 1. *There exists a constant C such that for every $a, b \geq 1$ if a graph has pathwidth less than a and no path of length 2^b, then the treedepth is bounded by Cab.*

Conjecture 2. *There exists a constant C such that for every $a, b \geq 1$ if a 3-connected graph has treewidth less than a and no path of length 2^b, then the treedepth is bounded by Cab.*

References

[1] K. Kawarabayashi and B. Rossman. A polynomial excluded-minor approximation of treedepth. In A. Czumaj, editor, *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 234–246. SIAM, 2018.

3 Shrub-depth vs long path as vertex minor

Hliněný, Kwon, Obdržálek, Ordyniak, Conjecture 6.3 of [1]

Prove of disprove: a class of graphs has bounded shrup-depth if and only if there exists an integer t such that no graph from the class contains a t-vertex path as a vertex-minor.

References

[1] P. Hlinený, O. Kwon, J. Obdrzálek, and S. Ordyniak. Tree-depth and vertex-minors. *Eur. J. Comb.*, 56:46–56, 2016.

4 Bicliques in complements of intersection graphs of curves

Pach and Tomon [1]

Prove or disprove: For every $\varepsilon > 0$ there are $c_0 = c_0(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ such that the complement of every intersection graph of curves, with $n \geq n_0$ vertices and at most $(\frac{1}{4} - \varepsilon)\binom{n}{2}$ edges, contains a bi-clique of size $c_0 n$, that is two disjoint equal sized sets of vertices with all the possible edges between them.

REFERENCES

[1] J. Pach and I. Tomon. Ordered graphs and large bi-cliques in intersection graphs of curves. *CoRR*, abs/1902.109810, 2019.

5 Computing chromatic number in subclasses of C_4-free graphs

Stemming from [1] and Huang's talk at CanaDAM 2019

Is CHROMATIC NUMBER polynomial on (a) $(C_4, 4P_1)$-free graphs? (b) (C_4, P_7)-free graphs? (c) (C_4, P_8)-free graphs. Gaspers, Huang, and Paulusma [1] proved that it is polynomial on (C_4, P_6)-free graphs by proving that every atom of such graph has bounded cliquewidth and showed NP-hardness proof for (C_4, P_9)-free graphs.

REFERENCES

[1] S. Gaspers, S. Huang, and D. Paulusma. Colouring square-free graphs without long induced paths. In R. Niedermeier and B. Vallée, editors, *35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France*, volume 96 of *LIPIcs*, pages 35:1–35:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018.

6 Hamilton paths in tripartite graphs

Yani Pehova

This is a problem about bipartite directed graphs. A directed graph has edges which are *ordered* pairs of distinct vertices, and for any vertices v_1, v_2, both (v_1, v_2) and (v_2, v_1) may be in the edge set. The *semidegree* of a vertex is the minimum of its in- and outdegree.

Proposition 3. *Let H be a balanced bipartite digraph on $2m$ vertices with minimum semidegree at least $(m+1)/2$, where m is large. Suppose x and y are two vertices in distinct vertex classes of H. Then H contains a directed Hamilton path from x to y.*

This proposition is not too difficult to prove by reducing the problem to a matching problem in an auxiliary bipartite graph that satisfies a similar minimum degree condition.
Building on this proposition (somewhat substantially), one may prove an approximate version of a conjecture due to Jackson which states that every regular bipartite tournament has a Hamilton decomposition.
We conjecture, or wish to disprove, the following:

Conjecture 4. *Let H be a tripartite balanced digraph on $3m$ vertices with minimum semi-degree $c'm$ for some $c' > 1$ and m large. Suppose x and y are two vertices in distinct vertex classes of H. Then H contains a directed Hamilton path from x to y.*
Update: *adding assumption that the in- and out-degrees are between $(1+\delta)m$ and $(1+2\delta)m$ for some small $\delta > 0$.*

Update 26.06 (Sebastian, Łukasz, Bartek K, Yani, Lena) Proved in quasirandom case when the in- and out- degree of v into V_i is at least $(1/2+\delta)m$. There is an example to the conjecture as above as stated, but in the example the degrees are uneven. Adding narrow brackets for the degrees may help.

7 Partitioning subcubic graphs

János Barát [1]

Barát conjectures that in every subcubic graph G on at least 7 vertices one can color the vertices red and blue such that the red subgraph induces a graph of maximum degree 1 and the blue subgraph induces a graph of minimum degree at least 1 and no 3-edge path. Barát proved the conjecture for trees and generalized Petersen graphs. It would be interesting to find more classes of subcubic graphs satisfying the conjecture.

Update 28.07 (Thomas, Terka, Martin, Marcin W, Lena) There is a infinite family of counterexamples both to (2-connected, planar, cubic) and (cubic, 3-connected) set of assumptions. The latter also disproves a conjecture of Thomassen [2].

References

[1] J. Barát. Decomposition of cubic graphs related to wegner's conjecture. *Discrete Mathematics*, 342(5):1520–1527, 2019.

[2] C. Thomassen. The square of a planar cubic graph is 7-colorable. *J. Comb. Theory, Ser. B*, 128:192–218, 2018.

8 Majority 3-colourings for tournaments

Carla Groenland

A *majority colouring* of a digraph if a function that assigns each vertex v a colour such that at most half of the out-neighbours of v receive the same colour as v. Kreutzer, Oum, Seymour, van der Zypen, and Wood [3] show that there exists a majority colouring with four colours and conjecture that it can be done with three colours. The four-coloring statement has been improved to four-choosability by Anholcer, Bosek, and Grytczuk [1]. Girão, Kittipassorn, and Popielarz [2] restrict to the special case of tournaments where they prove the conjecture if minimum degree is sufficiently large. They also show that such a colouring exists where at most a small constant number of vertices does not satisfy the desired condition. The existence of majority 3-colourings seems still open for tournaments and for general digraphs.

References

[1] M. Anholcer, B. Bosek, and J. Grytczuk. Majority choosability of digraphs. *Electr. J. Comb.*, 24(3):P3.57, 2017.

[2] A. Girão, T. Kittipassorn, and K. Popielarz. Generalized majority colourings of digraphs. *Combinatorics, Probability & Computing*, 26(6):850–855, 2017.

[3] S. Kreutzer, S. Oum, P. D. Seymour, D. van der Zypen, and D. R. Wood. Majority colourings of digraphs. *Electr. J. Comb.*, 24(2):P2.25, 2017.

9 Eulerian digraphs with no long path

Irene Muzi

(Solved) Is the following statement true? For every fixed t, the family of Eulerian digraphs without a path of length t is well-quasi ordered by the subgraph relation.

Update 27.07 (Archontia, Irene) A simple counterexample being a double-wheel.

10 Density of congested topological minors in sparse classes

Suggested by Michał Pilipczuk

(Resolved) We say that a graph H is a k-crossing topological minor of G if we can find a k-crossing topological minor model of H in G, defined as follows. Vertices of H are mapped to different vertices of G and edges of H are mapped to paths in G connecting corresponding images of endpoints so that: a path modelling an edge e does not traverse images of vertices of H other than the endpoints of e, and each path intersects at most k other paths. It is known that if G is planar, then the average degree of every its k-crossing topological minor is bounded by $O(\sqrt{k})$, see [1]. We conjecture that this is true even if G is K_t-minor-free for a fixed t, or even if G is K_t-topological-minor-free.

Update 26.06 (Piotr, Michał, Paweł, Bartek W, Marcin W) Proven for K_t-minor-free graphs.

References

[1] J. Pach and G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997.

11 $(14:5)$-coloring of subcubic triangle-free graphs

Suggested by Michał Pilipczuk

A graph G is $(a:b)$-colorable if given a palette of a colors, one can assign to every vertex b colors so that adjacent vertices receive disjoint subsets of colors. Clearly, if a graph is $(a:b)$-colorable, then its fractional chromatic number is at most $\frac{a}{b}$. Dvořák, Sereni, and Volec proved in [1] that every subcubic triangle-free graph has chromatic number at most $\frac{14}{5}$ by proving that it is $(14t:5t)$-colorable for some large t depending on the size of the graph. However, they conjecture that in fact every subcubic triangle-free graph is in fact $(14:5)$-colorable.

References

[1] Z. Dvorak, J. Sereni, and J. Volec. Subcubic triangle-free graphs have fractional chromatic number at most 14/5. *J. London Math. Society*, 89(3):641–662, 2014.

12 Chromatic number of triangle-free complements of string graphs

Bartosz Walczak

Estimate maximum possible chromatic number of triangle-free complements of string graphs.

13 Bounding treewidth/pathwidth/treedepth of P_t-free graphs in terms of largest biclique subgraph

Paweł Rzążewski

It is known that treedepth of a P_t-free graph is bounded linearly in its maximum degree. What about relaxing the maximum degree to maximum r such that $K_{r,r}$ is a subgraph of G? This is somewhat related [1].

References

[1] A. Atminas, V. V. Lozin, and I. Razgon. Linear time algorithm for computing a small biclique in graphs without long induced paths. In F. V. Fomin and P. Kaski, editors, *Algorithm Theory - SWAT 2012 - 13th Scandinavian*

Symposium and Workshops, Helsinki, Finland, July 4-6, 2012. Proceedings, volume 7357 of *Lecture Notes in Computer Science*, pages 142–152. Springer, 2012.

14 Large bicliques in intersection graphs of connected subgraphs of H-minor-free graphs

Paweł Rzążewski

For planar graphs, intersection graphs of connected subgraphs without $K_{t,t}$ as a subgraph have average degree $\mathcal{O}(t \log t)$. Is a similar statement true for H-minor-free graphs?

A FEW OPEN PROBLEMS FROM BARBADOS 2019 WORKSHOP

101 Long induced path in degenerate graphs

Louis Esperet, problem 2

Is the following statement true: For every integer k there exists a real $c > 0$ such that for every n and every k-degenerate graph G that contains a path of length, G contains an induced path of length $(\log n)^c$?
This is a question of Esperet, Lemoine, and Maffray [1].

References

[1] L. Esperet, L. Lemoine, and F. Maffray. Long induced paths in graphs. *Eur. J. Comb.*, 62:1–14, 2017.

102 Treewidth of theta- and triangle-free graphs

Nicolas Trotignon, problem 4

A *theta* is a graph consisting of three paths of length at least two each with common endpoints (and no other edges nor chords of the paths). Is it true that an n-vertex graph without an induced subgraph isomorphic to a theta nor a triangle has treewidth $\mathcal{O}(\log n)$?
One may think that the treewidth is bounded by a constant, but it is not the case, as shown by a construction due to Sintiari and Trotignon. Also, Radovanovic and Vuskovic proved that every graph with no theta and no triangle is the cube ($= K_{4,4}$ minus a matching), or has a clique cutset, or has a vertex of degree at most 2. This implies that they are all 3-colorable.

103 Coloring even-hole-free circular-arc graphs

Kristina Vuskovic, problem 16

What is the complexity of coloring even-hole-free circular-arc graphs?

104 Tree decomposition with tree being a subgraph

Zdeněk Dvořák, problem 26

Is it true that for every graph G of treewidth t there exists a tree decomposition (T, β) of G of with polynomial in t and T being a subgraph of G?

105 χ_3-boundedness of segment graphs

Bartosz Walczak, problem 30

We define $\chi_k(G)$ as the minimum number of colors in a coloring of G with no monochromatic clique of size k. We say a family of graphs $\mathcal{G}$ is χ_k-bounded if there exists a function f such that $\chi_k(G) \leq f(\omega(G))$ for each $G \in \mathcal{G}$.
Is the class of segment graphs (or 1-string graphs) χ_3 bounded?
This is false for string graphs. Krawczyk and Walczak showed an example with $\chi_k = \Theta((\log \log n)^{\omega-k+1})$.

106 χ-boundedness of string graphs in the upper plane

Rose McCarty, problem 31

Consider the class of string graphs drawn in the $y \geq 0$ plane where all strings intersect the line $y = 0$ and no string intersects itself. It has been shown that this class is χ-bounded by $\mathcal{O}(2^{\omega^2})$. Does this class polynomially χ-bounded?
James Davies and Rose McCarty recently showed that any circle graph with clique number ω has chromatic number at most $2\omega^2$, improving on the exponential bounds of Kostochka and Kratochvil [1]. They would like to know if the class of outerstring graphs, or even the class of intersection graphs of L-shapes, is χ-bounded by a polynomial. Both classes are χ-bounded and hereditary [3, 2], but the current bounds are exponential.

References

[1] A. Kostochka and J. Kratochvíl. Covering and coloring polygon-circle graphs. *Discrete Mathematics*, 163(1):299 – 305, 1997.

[2] S. McGuinness. On bounding the chromatic number of l-graphs. *Discrete Mathematics*, 154(1):179 – 187, 1996.

[3] A. Rok and B. Walczak. Outerstring graphs are χ-bounded. In S. Cheng and O. Devillers, editors, *30th Annual Symposium on Computational Geometry, SOCG'14, Kyoto, Japan, June 08 - 11, 2014*, page 136. ACM, 2014.

107 Strange orientation problem

Marthe Bonamy, Rose McCarty, problem 34

Does there exist a function f with the following property? Suppose G has an orientation D such that for all matchings M if S is the set of tails of M in D, then $\chi(G[S]) \leq d$. Then, $\chi(G) \leq f(d)$.

108 Cop numbers in hereditary graph classes

Vaidy Sivaraman (not on the list)

We are considering the classic cop-robber game. First, the cops player places cops on some vertices. Then, the robber chooses the starting vertex. Then, they alternate in moves, where each cop or robber can move over one edge. The cops win if a cop catches the robber by entering the same vertex. The cop number of a graph is a minimum number of cops required to catch the robber.
Prove the following conjectures (each later implies all previous):

- A $2K_2$-free graph G has cop number at most 2.
- A P_5-free graph G has cop number at most 2.
- A $\{C_6, C_7, C_8, ...\}$-free graph G has cop number at most 2.

109 Decreasing maximum average degree by deleting an independent set

Kevin Hendrey, Sergey Norin, David Wood, part of problem 14

(Solved) Maximum average degree of G, $\mathrm{mad}(G)$, is the maximum of the average degrees of subgraphs of G. Is it true that for every G there exists an independent set I in G with $\mathrm{mad}(G - I) \leq \mathrm{mad}(G) - 1$? It is even open with 1 replaced with an arbitrary positive absolute constant.

Update 27.07 (Wojtek, Marcin S, Bartek W) Solved: for every c, every graph of $\text{mad}(G) < c+1$ can be decomposed into $V(G) = V_1 \uplus V_2$ with V_1 independent and $\text{mad}(G[V_2]) < c$. (A more general claim that was here before turned out to have wrong proof, it is still open.)

110 Separation dimension and average degree

Alex Scott, David Wood, problem 9

Let G be a graph. An order π of $V(G)$ *separates* two disjoint edges e_1 and e_2 of G if the endpoints of e_1 appear before the endpoints of e_2 or vice versa. The *separation dimension* of G is the minimum cardinality of a set of orders of $V(G)$ such that every pair of disjoint edges of G is separated by at least one order. Is it true that graphs of separation dimension 4 have bounded average degree? Graphs of separation dimension 2 are planar and of separation dimension 3 have bounded average degree [1, 2].

References

[1] N. Alon, M. Basavaraju, L. S. Chandran, R. Mathew, and D. Rajendraprasad. Separation dimension and sparsity. *Journal of Graph Theory*, 89(1):14–25, 2018.

[2] A. Scott and D. Wood. Separation dimension and degree. *CoRR*, abs/1811.08994, 2018.

Change log

05 Sep 2019	more general claim for Problem 109 withdrawn
30 June 2019	final progress report
27 June 2019	progress report
26 June 2019	progress report
24 June 2019	two last-minute suggestions
20 June 2019	two suggestions by Michał
17 June 2019	one more reference for Problem 8
11 June 2019	suggestions by Carla and Irene
6 June 2019	Problem 7 suggested by Martin Merker
29 May 2019	Problem 6 by Yani Pehova
29 May 2019	two suggestions by Lena and one from CanaDAM
24 May 2019	a few more problems from Barbados, problem 2 and 3
30 Apr 2019	starting